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Proper Matter Collineations of Plane Symmetric Spacetimes

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We investigate matter collineations of plane symmetric spacetimes when the energy-momentum tensor is degenerate. There exists three interesting cases where the group of matter collineations is finite-dimensional. The matter collineations in these cases are either *four*, *six* or *ten* in which *four* are isometries and the rest are proper.

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Let (M, g) be a spacetime, where M is a smooth, connected, Hausdorff four-dimensional manifold and g is smooth Lorentzian metric of signature (+ - - -) defined on M. The manifold M and the metric g are assumed smooth (C^∞). A smooth vector field ξ is said to preserve a matter symmetry [1] on M if, for each smooth local diffeomorphism ϕ_t associated with ξ, the tensors T and $\phi_t^* T$ are equal on the domain U of ϕ_t, i.e., $T = \phi_t^* T$. Equivalently, a vector field ξ^a is said to generate a matter collineation if it satisfies the following equation

$$\pounds_\xi T_{ab} = 0, \quad or \quad T_{ab,c}\xi^c + T_{ac}\xi^c_{,b} + T_{cb}\xi^c_{,a} = 0, \quad (a, b, c = 0, 1, 2, 3), \qquad (1)$$

where $\pounds$ is the Lie derivative operator, ξ^a is the symmetry or collineation

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vector. Every Killing vector (KV) is an MC but the converse is not true, in general. A proper MC is an MC which is not a KV, or a homothetic vector.

A large body of literature exists on classification of spacetimes according to their isometries or Killing vectors and the groups admitted by them [2-5]. These investigations of symmetries played an important role in the classification of spacetimes, giving rise to many interesting results with useful applications. As curvature and Ricci tensors play a significant role in understanding the geometric structure of metrics, the energy-momentum tensor enables us to understand the physical structure of spacetimes. Some recent investigations [6-19] show keen interest in the study of matter collineations (MCs). In the papers [15-19], the study of MCs has been taken for spherically symmetric, static plane symmetric and cylindrically symmetric spacetimes and some interesting results have been obtained. In this letter, we find the proper MCs of non-static plane symmetric spacetimes when the energy-momentum tensor is degenerate. It turns out that this admits an MC Lie algebra of 4, 6, 10 dimensions apart from the infinite dimensional algebras.

The most general form of the plane symmetric metric is given in the form [4]

$$ds^2 = e^{\nu(t,x)}dt^2 - e^{\lambda(t,x)}dx^2 - e^{\mu(t,x)}(dy^2 + dz^2), \qquad (2)$$

where $\nu,\ \lambda$ and μ are arbitrary functions of t and x. The surviving components of the energy-momentum tensor are given as

$$\begin{aligned}
T_{00} &= \frac{1}{4}(\dot{\mu}^2 + 2\dot{\mu}\dot{\lambda}) - \frac{1}{4}e^{v-\lambda}(4\mu'' + 3\mu'^2 - 2\mu'\lambda'),\\
T_{01} &= -\frac{1}{2}(2\dot{\mu}' + \dot{\mu}\mu' - \dot{\mu}v' - \mu'\dot{\lambda}),\\
T_{11} &= \frac{1}{4}(\mu'^2 + 2\mu'\nu') - \frac{1}{4}e^{\lambda-\nu}(4\ddot{\mu} + 3\dot{\mu}^2 - 2\dot{\mu}\dot{\nu}),\\
T_{22} &= \frac{1}{4}e^{\mu-\nu}(2\mu'' + \mu'^2 - \mu'\lambda' + \mu'\nu' - \lambda'\nu' + \nu'^2 + 2\nu'')\\
&- \frac{1}{4}e^{\mu-\nu}(2\ddot{\mu} + \dot{\mu}^2 - \dot{\mu}\dot{\nu} + \dot{\mu}\dot{\lambda} - \dot{\nu}\dot{\lambda} + \dot{\lambda}^2 + 2\ddot{\lambda}),\\
T_{33} &= T_{22}. \qquad (3)
\end{aligned}$$

The MC equations can be written as follows

$$\begin{aligned}
2T_{01}\xi^1_{,0} + T_{00,0}\xi^0 + T_{00,1}\xi^1 + 2T_{00}\xi^0_{,0} &= 0, \qquad (4)\\
T_{01,0}\xi^0 + T_{01}\xi^0_0 + T_{00}\xi^0_{,1} + T_{01,1}\xi^1 + T_{11}\xi^1_{,0} + T_{01}\xi^1_{,1} &= 0, \qquad (5)
\end{aligned}$$

$$
\begin{aligned}
T_{00}\xi^0_{,2} + T_{01}\xi^1_{,2} + T_{22}\xi^2_{,0} &= 0, && (6)\\
T_{00}\xi^0_{,3} + T_{01}\xi^1_{,3} + T_{22}\xi^3_{,0} &= 0, && (7)\\
T_{11,0}\xi^0 + 2T_{01}\xi^0_1 + T_{11,1}\xi^1 + 2T_{11}\xi^1_{,1} &= 0, && (8)\\
T_{01}\xi^0_{,2} + T_{11}\xi^1_{,2} + T_{22}\xi^2_{,1} &= 0, && (9)\\
T_{01}\xi^0_{,3} + T_{11}\xi^1_{,3} + T_{22}\xi^3_{,1} &= 0, && (10)\\
T_{22,0}\xi^0 + T_{22,1}\xi^1 + 2T_{22}\xi^2_{,2} &= 0, && (11)\\
T_{22}(\xi^2_{,3} + \xi^3_{,2}) &= 0, && (12)\\
T_{22,0}\xi^0 + T_{22,1}\xi^1 + 2T_{22}\xi^3_{,3} &= 0. && (13)
\end{aligned}
$$

These are the first order non-linear partial differential equations in four variables $\xi^a(x^b)$. We solve these equations for the degenerate case, where $\det(T_{ab}) = 0$ and also we restrict ourselves with the assumption $T_{01} = 0$. For the sake of simplicity, we use the notation $T_{aa} = T_a$. The following two cases satisfy these assumptions:

(I) $SO(3) \otimes \mathbf{R}$, where $\mathbf{R} = \partial_t$ if and only if
(a) $\nu = \nu(x),\ \lambda = \lambda(x),\ \mu = 2\ln x$ or
(b) $\nu = \nu(x),\ \lambda = 0,\ \mu = 2\ln a$,
where a is an arbitrary constant.

(II) $SO(3) \otimes \mathbf{R}$, where $\mathbf{R} = \partial_x$ if and only if
(a) $\nu = \nu(t),\ \lambda = \lambda(t),\ \mu = 2\ln t$ or
(b) $\nu = 0,\ \lambda = \lambda(t),\ \mu = 2\ln a$.

CASE I

This case has the following two possibilities:
(Ia) $\nu = \nu(x),\ \lambda = \lambda(x),\ \mu = 2\ln x$,
(Ib) $\nu = \nu(x),\ \lambda = 0,\ \mu = 2\ln a$.
Case (Ia): Here we solve MC Eqs.(4)-(13), when at least one of $T_a = 0$. This can be classified in the following three main cases:

(1) When only one of $T_a \neq 0$,

(2) When two of $T_a \neq 0$,

(3) When three of $T_a \neq 0$.

Case (1): In this case, there could be further two possibilities:

(1a) $T_0 \neq 0, \quad T_i = 0, \quad (i = 1, 2, 3),$

(1b) $T_1 \neq 0, \quad T_j = 0, \quad (j = 0, 2, 3).$

When we solve MC equations of the case 1, we obtain infinite dimensional MCs for all the possibilities.

Case (2): This case admits the following two subcases:

(2a) $T_\ell = 0, \quad T_k \neq 0, \quad (\ell = 0, 1),$

(2b) $T_\ell \neq 0, \quad T_k = 0, \quad (k = 2, 3).$

After some algebra, we arrive at the conclusion that this case also leads to infinite dimensional MCs in all the options.

Case (3): This is an interesting case of the degenerate energy-momentum tensor which gives some finite dimensional MCs. Here we take only one component of the energy-momentum tensor equals to zero. The two possibilities are the following:

(3a) $T_0 = 0, \quad T_i \neq 0, \quad (i = 1, 2, 3).$

(3b) $T_1 = 0, \quad T_j \neq 0, \quad (j = 0, 2, 3).$

Case (3a): All the possibilities arising from this case yield infinite dimensional MCs.

Case (3b): This case explores further four possibilities:

(i) $T_0 = constant \neq 0, \quad T_2 = constant \neq 0,$

(ii) $T_{0,1} \neq 0, \quad T_2 = constant \neq 0,$

(iii) $T_0 = constant \neq 0, \quad T_{2,1} \neq 0,$

(iv) $T_{0,1} \neq 0, \quad T_{2,1} \neq 0.$

The cases 3b(i-iii) yield infinite dimensional MCs but the case 3b(iv) gives finite dimensional MCs. When we solve the MC equations under the assumptions of this case, we further obtain the following three options:

$$(*) \qquad \frac{T_{0,1}T_2}{T_{2,1}T_0} = \varepsilon \neq 0, \quad \left(\frac{T_0}{T_2}\right)' \neq 0,$$
$$(**) \qquad T_0 = mT_2, \quad (***) \quad \left(\frac{T_{0,1}T_2}{T_{2,1}T_0}\right)' \neq 0,$$

where m is an arbitrary constant. In the first option, MCs turn out to be

$$\begin{aligned} \xi_{(1)} &= \partial_t, \quad \xi_{(2)} = \partial_y, \quad \xi_{(3)} = \partial_z, \quad \xi_{(4)} = z\partial_y - y\partial_z, \\ \xi_{(5)} &= t\partial_t - \frac{2T_0}{T_{0,1}}\partial_x, \quad \xi_{(6)} = \iota(z\partial_y - y\partial_z). \end{aligned} \tag{14}$$

This gives six independent MCs out of which two are proper MCs. The Lie algebra has the following commutators:

$$[\xi_{(1)}, \xi_{(5)}] = \xi_{(1)}, \quad [\xi_{(2)}, \xi_{(4)}] = -\xi_{(3)}, \quad [\xi_{(2)}, \xi_{(6)}] = -\iota\xi_{(3)},$$

$$[\xi_{(3)}, \xi_{(4)}] = \xi_{(2)}, \quad [\xi_{(3)}, \xi_{(6)}] = \iota\xi_{(2)}, \quad [\xi_{(i)}, \xi_{(j)}] = 0, \ \text{(otherwise)}.$$

For the second possibility, we obtain six proper MCs given by

$$\begin{aligned} \xi_{(5)} &= \frac{1}{m}\left[ty\partial_t - \frac{2T_2}{T_{2,1}}y\partial_x + \frac{1}{2}(y^2 - z^2 - mt^2)\partial_y + yz\partial_z\right], \\ \xi_{(6)} &= tz\partial_t - \frac{2T_2}{T_{2,1}}z\partial_x + yz\partial_y - \frac{1}{2}(y^2 - z^2 - mt^2)\partial_z, \\ \xi_{(7)} &= t\partial_t - \frac{2T_2}{T_{2,1}}\partial_x + y\partial_y + z\partial_z, \\ \xi_{(8)} &= \frac{1}{2m}(y^2 + z^2 - mt^2)\partial_t + \frac{2T_2}{T_{2,1}}t\partial_x - ty\partial_y - tz\partial_z, \\ \xi_{(9)} &= \frac{1}{m}y\partial_t - t\partial_y, \quad \xi_{(10)} = \frac{1}{m}z\partial_t - t\partial_z. \end{aligned} \tag{15}$$

The Lie algebra is given by

$$[\xi_{(1)}, \xi_{(5)}] = \xi_{(9)}, \quad [\xi_{(1)}, \xi_{(6)}] = m\xi_{(10)}, \quad [\xi_{(1)}, \xi_{(7)}] = \xi_{(1)},$$

$$[\xi_{(1)},\xi_{(8)}]=-\xi_{(7)},\quad [\xi_{(1)},\xi_{(9)}]=-\xi_{(2)},\quad [\xi_{(1)},\xi_{(10)}]=-\xi_{(3)},$$

$$[\xi_{(2)},\xi_{(4)}]=-\xi_{(3)},\quad [\xi_{(2)},\xi_{(5)}]=\frac{1}{m}\xi_{(7)},\quad [\xi_{(2)},\xi_{(6)}]=\xi_{(4)},$$

$$[\xi_{(2)},\xi_{(7)}]=\xi_{(2)},\quad [\xi_{(2)},\xi_{(8)}]=\xi_{(9)},\quad [\xi_{(2)},\xi_{(9)}]=\frac{1}{m}\xi_{(1)},$$

$$[\xi_{(3)},\xi_{(4)}]=\xi_{(2)},\quad [\xi_{(3)},\xi_{(5)}]=-\frac{1}{m}\xi_{(4)},\quad [\xi_{(3)},\xi_{(6)}]=\xi_{(7)},$$

$$[\xi_{(3)},\xi_{(7)}]=\xi_{(3)},\quad [\xi_{(3)},\xi_{(8)}]=\xi_{(10)},\quad [\xi_{(3)},\xi_{(10)}]=\frac{1}{m}\xi_{(1)},$$

$$[\xi_{(4)},\xi_{(5)}]=\frac{1}{m}\xi_{(6)},\quad [\xi_{(4)},\xi_{(6)}]=-m\xi_{(5)},\quad [\xi_{(4)},\xi_{(7)}]=\xi_{(4)},$$

$$[\xi_{(4)},\xi_{(10)}]=-\xi_{(9)},\quad [\xi_{(5)},\xi_{(7)}]=\xi_{(5)},\quad [\xi_{(5)},\xi_{(9)}]=-\frac{1}{m}\xi_{(8)},$$

$$[\xi_{(6)},\xi_{(7)}]=\xi_{(6)},\quad [\xi_{(6)},\xi_{(10)}]=-\xi_{(8)},\quad [\xi_{(7)},\xi_{(8)}]=\xi_{(8)},$$

$$[\xi_{(8)},\xi_{(9)}]=\xi_{(5)},\quad [\xi_{(8)},\xi_{(10)}]=\frac{1}{m}\xi_{(6)},\quad [\xi_{(9)},\xi_{(10)}]=\frac{1}{m}\xi_{(4)},$$

$$[\xi_{(i)},\xi_{(j)}] \;=\; 0, \quad \text{(otherwise)}.$$

The last option yields four independent MCs which are the usual KVs of the static plane symmetry.

CASE II

This case has two subcases
(a) $\nu = \nu(t),\ \lambda = \lambda(t),\ \mu = 2\ln t$,
(b) $\nu = 0,\ \lambda = \lambda(t),\ \mu = 2\ln a$.

Case(IIa): The other possibilities can be classified in the following three main cases:

(1) When only one of $T_a \neq 0$,

(2) When two of $T_a \neq 0$,

(3) When three of $T_a \neq 0$.

The cases (1) and (2) lead to infinite dimensional MCs in all the possibilities.

Case (3): This case of the degenerate energy-momentum tensor yields some finite dimensional MCs. The following two possibilities arise:

(3a) $T_0 = 0, \quad T_i \neq 0, \quad (i = 1, 2, 3)$.

(3b) $T_1 = 0, \quad T_j \neq 0, \quad (j = 1, 2, 3)$.

Case (3a): This case further leads to the following four possibilities:

(i) $T_1 = constant \neq 0, \quad T_2 = constant \neq 0$,

(ii) $T_{1,0} \neq 0, \quad T_2 = constant \neq 0$,

(iii) $T_1 = constant \neq 0, \quad T_{2,0} \neq 0$,

(iv) $T_{1,0} \neq 0, \quad T_{2,0} \neq 0$.

The cases 3a(i) and 3a(ii) yield the infinite dimensional MCs.

Case (3aiii): When we solve MC equations simultaneously for this case, we obtain the following finite dimensional MCs

$$\begin{aligned} \xi_{(1)} &= \partial_x, \quad \xi_{(2)} = \partial_y, \quad \xi_{(3)} = \partial_z, \quad \xi_{(4)} = z\partial_y - y\partial_z, \\ \xi_{(5)} &= y\partial_x - mx\partial_y, \quad \xi_{(6)} = z\partial_x - mx\partial_z. \end{aligned} \tag{16}$$

We obtain six independent MCs. The corresponding Lie algebra has the following commutators:

$$[\xi_{(1)},\xi_{(6)}]=-m\xi_{(3)},\quad [\xi_{(1)},\xi_{(5)}]=-m\xi_{(2)},\quad [\xi_{(6)},\xi_{(5)}]=-m\xi_{(4)},$$

$$[\xi_{(6)},\xi_{(4)}]=-\xi_{(5)},\quad [\xi_{(6)},\xi_{(3)}]=-\xi_{(1)},\quad [\xi_{(5)},\xi_{(4)}]=-\xi_{(6)},$$

$$[\xi_{(5)},\xi_{(2)}]=-\xi_{(1)},\quad [\xi_{(4)},\xi_{(2)}]=\xi_{(3)},\quad [\xi_{(4)},\xi_{(3)}]=-\xi_{(2)},$$

$$[\xi_{(i)},\xi_{(j)}]\;=\;0,\quad \text{(otherwise)}.$$

Case (3aiv): This case turns out exactly the same as (3aiii).
Case (3b): This case leads to infinite dimensional MCs.
Case (IIb): It is similar to the case II(2a).

In a recent paper [19], some interesting results have been obtained when we classify static plane symmetric spacetimes according to their energy-momentum tensor. This idea has been used to find the proper MCs of non-static plane symmetric spacetimes for the degenerate case only. It is worth mentioning that we have found four such cases having either four, six or ten independent MCs. The results are summarized in the form of table 1.

Table 1. MCs for the Degenerate Case (only finite cases)

Cases	**MCs**	**Constraints**
I3biv(*)	6	$T_1=0,\ T_j\neq 0(j=0,2,3),\ T_2'\neq 0,\ \frac{T_0'T_2}{T_0T_2'}\neq 0,\ (\frac{T_0}{T_2})'\neq 0$
I3biv(**)	10	$T_1=0,\ T_j\neq 0,\ T_2'\neq 0,\ T_0=mT_2,\ T_0'\neq 0$
I3biv(***)	4	$T_1=0,\ T_j\neq 0,\ T_2'\neq 0,\ \frac{T_0'T_2}{T_0T_2'}\neq 0$
II3a(iii)	6	$T_0=0,\ T_{1,0}\neq 0,\ T_{2,0}\neq 0,\ T_1=mT_2$

This shows that each case has different constraints on the energy-momentum tensor. When the rank of T_a is 3, i.e. $T_1=0$, we obtain the following metric

$$ds^2=e^{\nu}dt^2-dx^2-e^{-2\nu}(dy^2+dz^2), \tag{17}$$

where ν is an arbitrary function of x only. It can be easily verified that this class of metrics represent perfect fluid dust solutions. The energy-density for the above metrics is given as

$$\rho=(2\nu''-3\nu'^2)e^{\frac{\nu}{2}}, \tag{18}$$

We can conclude that when the rank of T_{ab} is 1 or 2, all the possibilities yield infinite dimensional MCs. If the rank of T_{ab} is 3, this leads to some possibilities of finite dimensional MCs. It would be interesting to classify plane symmetric spacetimes according to their MCs for the non-degenerate case and then removing the assumption $T_{01} = 0$. All these problems would be investigated as a separate issue.

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